RESISTANCE, REACTANCE AND IMPEDANCE
A Primer

Douglas Brooks, PhD
UltraCAD Design, Inc.

PART 2, REACTANCE

This is part 2 of a 3-part series on resistance, reactance and impedance. Most of us are familiar with resistance --- at least we think we are. Those will find Part 1 pretty basic. But few of us really understand reactance and its relationship to resistance. Part 2 gets a lot more interesting! And while most of us sort of understand what impedance is, few really understand the relationship between it and the other two properties. This 3-part series will tie them all together.

Reactance:

Resistance and reactance are dramatically different. Resistance does not depend on frequency, reactance does. Resistance does not cause a phase shift, reactance causes a 90 degree phase shift between voltage and current. Resistance causes the loss of (i.e. dissipates) power, reactance does not. Pure (ideal) reactance returns all energy that it stores in its field.

Reactance is what we get with capacitors and inductors. But capacitors and inductors are almost exactly opposite in their effects. Both "impede" current. But capacitance impedes current at low frequencies, inductance at high frequencies. Capacitance causes voltage to *lag* current by 90 degrees (thus we say there is a -90 degree phase shift) while inductance causes voltage to *lead* current by 90 degrees (thus we say that there is a +90 degree phase shift.) Capacitors store energy in the electric field between their plates, inductors store energy in the magnetic field around the wire.

The three passive components in electronics, resistors, capacitors, and inductors act as though they are on a spectrum. Capacitors and inductors are at opposite ends of the spectrum, resistors are at the exact center of the spectrum. (See Brooks, Note 1.)

Capacitance:

In simple terms a capacitor is formed by two (or more) parallel plates separated by a material we call a dielectric. In an electronic circuit (the only case of interest to us in this article series) when charge (current) flows onto one of the plates, an equal amount of charge flows off the other plate. Thus, there has to be a current loop for current to flow, satisfying the first of the three current laws we talked about in Part 1.

Since there is a separation between the two plates, the electrons themselves can't cross over between the plates. Nevertheless, we have current flow because the definition of current is the flow of electrons. When electrons flow onto one plate, they flow off the other. There is current flow around the loop. Maxwell had a problem visualizing this, so he modeled the "apparent" flow across the void between the plates, calling it "displacement" current. (See Brooks, Note 2.)

A capacitor does not have to have parallel plates. There can be (and is) capacitance between a trace and a plane, between two wires or two traces, and between a wire and the Earth ground (with air as the dielectric in between.)

When current (charge) flows onto one plate (and therefore off the other plate) a charge *differential* exists across the plates. This charge differential, by definition, is voltage (see Part 1 of this series.) This voltage between the plates causes a *charge field*, or an *electric field* to exist between the plates. It takes energy to create this field. That is, it took some energy to push the electrons onto one plate and off the other plate. That amount of energy (and *exactly* that amount of energy!) is stored in the electric field. The voltage across the plates is a direct measure of that energy. If current (charge) is allowed to flow back around the plates so that there is no net charge between the plates, all the stored energy is recovered by the system and there is no net energy loss in the system.

Phase Shift of a Capacitor:

If current flows onto the plates of a capacitor, the voltage changes slowly. The larger the capacitor, the more slowly the voltage changes --- i.e. it takes more charge to cause a given change in voltage. So we see that current has to flow *first* before voltage changes. That means that current *leads* voltage, or voltage *lags* current. This is what is meant by phase shift. In capacitors, current leads voltage.

Measure of Capacitance:

The measure of capacitance is the Farad, named after Michael Faraday (see Note 3). If one coulomb of charge (remember one coulomb of charge is 6.25×10^{18} electrons) flowing onto the plates of a capacitor results in a voltage across the plates of one volt, we have one farad of capacitance. One farad of capacitance is pretty big. On circuit boards we typically see capacitors measured in "micro-farads" (uf, or 10^{-6} farads) or "pico-farads" (pf or 10^{-12} farads).

A larger capacitor can store more charge at the same voltage. Capacitance is directly proportional to the parallel cross sectional area and inversely proportional to the distance between those plates. It is also directly proportional to the relative dielectric coefficient of the material between the plates. So to increase the capacitance between two conductors, increase the cross sectional area of the facing surfaces, decrease the distance between them, and increase the relative dielectric coefficient of the material between them (i.e. use a different material.) To reduce capacitance between two conductors, place them further apart, reduce the cross sectional area of the facing surfaces, and reduce the relative dielectric coefficient of the material between the surfaces.

Inductance:

In its simplest form, an inductor is a coiled wire. But it does not have to be a coil. Even a straight length of wire has inductance.

The nature of inductance is a changing magnetic field. Most of us are aware that if a current flows down a wire or a trace there is a magnetic field around that wire or trace. That is the fundamental principle behind an electromagnet. So, if the current changes, so too does the magnetic field. That's where it gets interesting. Faraday's Law says that a *changing magnetic field around a wire or trace induces a current in that wire or trace in the opposite direction to the current causing the changing magnetic field*. This is the fundamental principal behind a transformer. A *changing* current in the primary winding induces a current in the secondary winding (note the emphasis on *changing*.) It is also the fundamental principal behind antenna transmissions, EMI, and cross talk.

So consider a *change* in the current flowing down a wire. That causes a *change* in the magnetic field around the wire. That *change* in magnetic field induces a current *in that same wire* in the *opposite* direction to the original change in current flow. In the first instant of time, these two currents are equal and opposite and they cancel. So there is no net change in current flow. In the next instant of time the magnetic field weakens slightly and a little current flows. In the next instant of time, a little more current flows. At some point, the full current flows down the wire. What I have just described is the effect of inductance on current flow.

It takes energy to create the magnetic field around a wire. That amount of energy (and exactly that amount of energy) is stored in the magnet field. If current were to stop flowing, the magnetic field would collapse. By definition, while it is collapsing, it is changing. That changing (collapsing) magnetic field returns energy to the circuit. So in the end, there is no energy lost to the system.

Measure of Inductance:

The measure of inductance is the "henry" (see Note 4.) The formal definition of a henry is a little obscure and involves magnetic field strength, but an intuitive definition goes like this: *One henry of inductance will allow one amp of current per second to flow under the force of one volt.* A henry is a pretty big inductance. On circuit boards we typically work with inductors and inductances measured in milli-henry (mH or 10^{-3}H) and nanohenry (nH or 10^{-9} H).

A larger inductance offers more *impedance* to current flow (at the same frequency) than does a smaller inductance. We make inductance larger by increasing the density of the magnet field. We can accomplish this in several ways:
1. Reduce the size of the wire or trace (thereby increasing the density of the field around the wire.) This is why wider traces have less inductance and planes have fairly low inductance.
2. Coil the wire. Each turn of the coil adds one more increment of magnetic field around the wire.
3. Add the presence of iron (ferrite) near the wire. A ferrite bead around a wire increases the intensity (density) of the magnetic field around the wire at that point.
(See Note 5.)

Phase Shift of Inductance:

If the voltage across an inductor changes suddenly, at the first instant of time no current flows. This is the nature of inductance. That means that current follows *behind* voltage, or current *lags* voltage. Alternatively voltage *leads* current through an inductor. This is what is meant by the phase shift of an inductor, and it is exactly the opposite of what occurs with a capacitor.

The Imaginary Operator:

There comes a point in the basic class in electrical engineering where the professor goes through a mathematic proof and ends up with a mathematical expression for the impedance of a circuit that looks something like this (Z, R and X are variables that may include several terms):
$$Z = R + \sqrt{-1}*X \hspace{4cm} \text{Equation 1}$$
Then there is a pregnant pause while the professor waits for the individuals in the class to "catch up" with what he has just written. And what he has just written includes a term (denoted

by X) multiplied by the square root of minus one. The problem? There is no such thing as the square root of minus one. There is no number, multiplied by itself, that equals -1.

The general expression for such an equation is Z = R + iX, where i stands for $\sqrt{-1}$ and is called the imaginary operator. This area of math is called complex algebra (see Brooks, Note 6). The fact that the square root of minus one can show up in a mathematical proof has been know as far back as ancient times (see Note 7). Throughout history there have been various reactions to it. In ancient times they ignored it as impossible. In later times it was considered demonic and/or satanic and people who tried to use it might have been burned at the stake. In still later times we finally perceived more rational explanations.

In electronics we do not use "i" to denote the imaginary operator because it can get confused for the symbol, i, we use for current. So we use "j" for the imaginary operator. So the general equation for impedance becomes Z = R + jX, where R denotes the term for resistance and X denotes the term for reactance. And here is the practical interpretation for j in electronics: *The imaginary operator means a 90 degree phase shift.* So, looking at the general impedance equation we have a term for resistance (R, Part 1) with zero phase shift and a term for reactance (X, Part 2). The j means that the reactance has a 90 degree phase shift with respect to resistance.

Reactance of a Capacitor and Inductor:

The term for resistance (Part 1) was pretty simple. It contained one variable, the value of the component R. The term for reactance is more complicated. It contains two variables and a constant. The first variable is the value of the component (C or L for capacitance or inductance). The second variable represents frequency, and the constant is the imaginary operator.

But it gets worse (at least for some of us.) Solutions to electronic equations involve calculus. There is no escape from that. EEs get an unbelievable amount of calculus in school. The transforms used in the calculus in electronics result in trigonometric equations, which is where all the sines, cosines, and phase angles come from. And the angles for all those trigonometric equations are measured in radians (not degrees!) Those of us raised up being comfortable with the fact that there are 360 degrees in a circle (or one cycle) have to adjust to radians (easier for some of us than for others.) The conversion is that there are 2pi (2π) radians in 360 degrees or in one cycle.

So in electronics we typically refer to 360 degrees as one cycle. We let f represent frequency in cycles, so 360 degrees would equal one cycle, 720 degrees two cycles, etc. The historical measure of frequency was cycles per second (f), changed to hertz (Hz) in 1960 (See Note 8). But in the mathematics of electronics we have to use the *angular frequency* (ω) in radians, in which:

$$\omega = 2\pi f \hspace{3cm} \text{Equation 2}$$

So now we are ready. The reactance of an inductor and a capacitor, respectively, are:

$$X_L = j\omega L = j2\pi L \hspace{1cm} \text{(Ohms)} \hspace{2cm} \text{Equation 3}$$

$$X_C = \frac{1}{j\omega C} = \frac{1}{j2\pi fC} \hspace{1cm} \text{(Ohms)} \hspace{2cm} \text{Equation 4}$$

Recognizing that the standard form for the reactance term in an equation is jX, let's get the j term in the numerator in Equation 4 by multiplying both numerator and denominator by j (note that the j^2 term in the denominator becomes -1):

$$X_C = \frac{-j}{\omega C} = \frac{-j}{2\pi fC} \qquad \text{Equation 5}$$

Now, the j with the (implied) plus sign in equation 3 means + 90 degree phase shift (voltage leads current), the j with the minus sign in Equation 5 means a minus 90 degree phase shift (voltage lags current).

Example: Suppose we want to know the reactance of a 5 uH inductor and/or a 0.1 uF capacitor at a frequency (f) of 1.0 Mhz.

$$X_C = \frac{-j}{2\pi 10^6 (0.1)10^{-6}} = -j1.59\Omega \qquad \text{Equation 6}$$

$$X_L = j2\pi 10^6 5(10^{-6}) = j31.4\Omega \qquad \text{Equation 7}$$

The -j term in Equation 6 does not mean a negative reactance in Ohms. It means that the reactance is 1.59 Ohms with a -90 degree phase shift. This is an important difference. In general, reactance cannot be negative, only the phase shift can. The (+)j term in Equation 7 means a plus 90 degree phase shift.
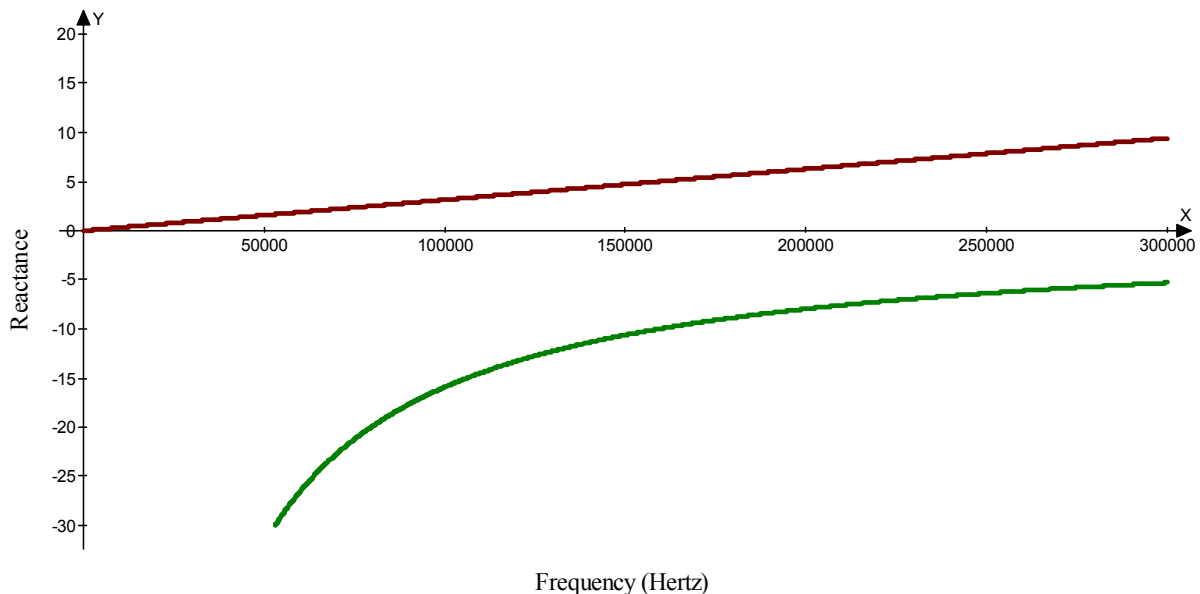

Figure 1
Plot of Equation 6 (X$_C$, in green) and Equation 7 (X$_L$, in red.)

Figure 1 is a plot of Equation 6 and 7. While it is correct, it is not the conventional way we graph reactance. A more conventional plot is shown in Figure 2.
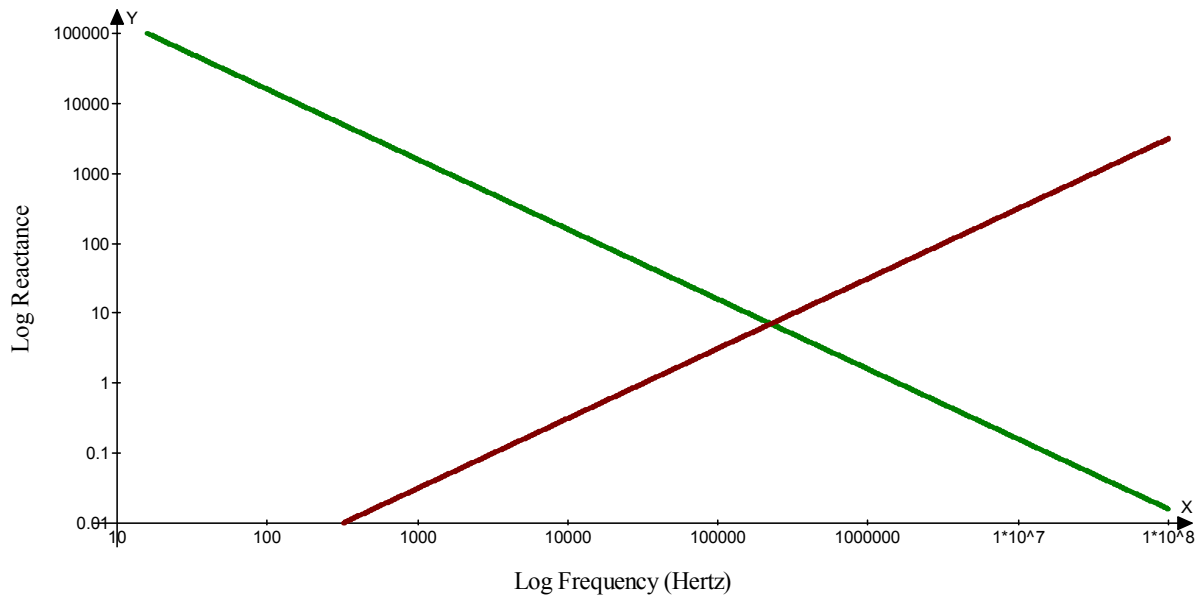

Figure 2
A more conventional plot of Equation 6 (X_C, in green) and Equation 7 (X_L, in red.)

There are two important differences in Figure 2 compared to Figure 1:
   a. The axes are logarithmic. This is because the range of values for reactance and frequency can cover a very wide range. Data can be visualized more meaningfully using logarithmic axes.
   b. The absolute value (without regard to sign) of the reactance is plotted. The reason for doing this will become very clear in Part 3. For now, consider the fact that it is not reactance that is negative. In fact, for all practical purposes, reactance can't be negative. It is the phase shift that is negative.

Now, looking at Figure 2 we see that the curve for capacitive reactance is high at lower frequencies and lower at higher frequencies. Thus, we say that the curve for capacitive reactance is "downward sloping to the right." The curve for inductive reactance, on the other hand, is "upward sloping to the right." This will *always* be true! In any plot of reactance vs. frequency, any *part* of the curve that is downward sloping to the right will be capacitive, and will, therefore, have a phase shift of -90 degrees over that region. Any *part* of a reactance curve vs. frequency that is upward sloping to the right will be inductive, and will therefore have a phase shift of +90 degrees over that region.

(Note: Any part of a reactance curve that is horizontal vs. frequency, i.e. constant reactance independent of frequency, would be resistive and have zero phase shift, although such a region might be unlikely to occur except under very special circumstances. When we talk about impedance (as opposed to reactance) in Part 3 we will make very similar comments about the

impedance curve vs. frequency, except that other phase shifts between -90 degrees and +90 degrees will be possible.)

Series and Parallel Combinations of Reactance:

Reactances combine in series and parallel combinations just like resistance does. We can work out the equivalent reactance for each case using Kirchhoff's second and first laws, respectively, just as we did for resistance in Part 1 of this series. So referencing Figure 3, we can derive that:
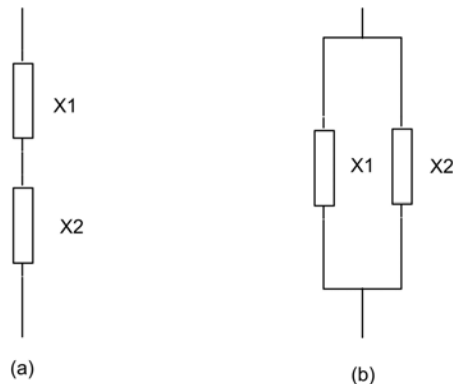

(a)   (b)

Figure 3
Series and parallel combinations of reactance.

For series combinations: X_{series} = X1 + X2   Equation 8

For parallel combinations: $X_{parallel} = \dfrac{1}{\dfrac{1}{X1} + \dfrac{1}{X2}}$ Equation 9

Capacitors:

Plugging in the formulas for X_C from Equation 5, we see that series capacitors combine:

$$X_{Series} = \frac{1}{\omega C_{eq}} = \frac{1}{\omega C1} + \frac{1}{wC2}$$ Equation 10

From which it follows that (for series combinations):

$$C_{eq} = \frac{1}{\dfrac{1}{C1} + \dfrac{1}{C2}}$$ Equation 11

Capacitors in parallel combine as:

$$X_{Parallel} = \frac{1}{\omega C_{eq}} = \frac{1}{\dfrac{1}{1/\omega C1} + \dfrac{1}{1/\omega C2}} = \frac{1}{\omega C1 + \omega C2}$$

Equation 12

From which it follows that (for parallel combinations)

$$C_{eq} = C1 + C2$$

Equation 13

Inductors:

At this point you have enough information to figure out that inductors combine just like resistors do:

For series combinations:

$$L_{eq} = L1 + L2$$

Equation 14

For parallel combinations:

$$L_{eq} = \frac{1}{\dfrac{1}{L1} + \dfrac{1}{L2}}$$

Equation 15

Capacitor and Inductor:

There is a very interesting case of what happens when we combine a capacitor and an inductor in series (or in parallel). Such a combination occurs routinely on a circuit board with a real-world capacitor with some lead/pad inductance.

The series reactance in such a case is given by Equation 8 with the appropriate values plugged in.

$$X_{Series} = j\omega L + \frac{1}{j\omega C} = j\left(\omega L - \frac{1}{\omega C}\right) = j\left(\frac{\omega^2 LC - 1}{\omega C}\right)$$

Equation 16

If we place a capacitor and an inductor in parallel, and use Equation 9, we get:

$$X_{parallel} = \frac{1}{\dfrac{1}{j\omega L} + \dfrac{1}{1/j\omega C}} = \frac{j\omega L}{1 - \omega^2 LC}$$

Equation 17

Using the values for L and C that were used in Figure 2, the reactance curves for these equations are plotted in Figure 4. As before, the green curve is the capacitor and the red curve

is the inductor. The new, blue, curve is the series combination of the two, and the dotted blue curve is the parallel combination of the two.
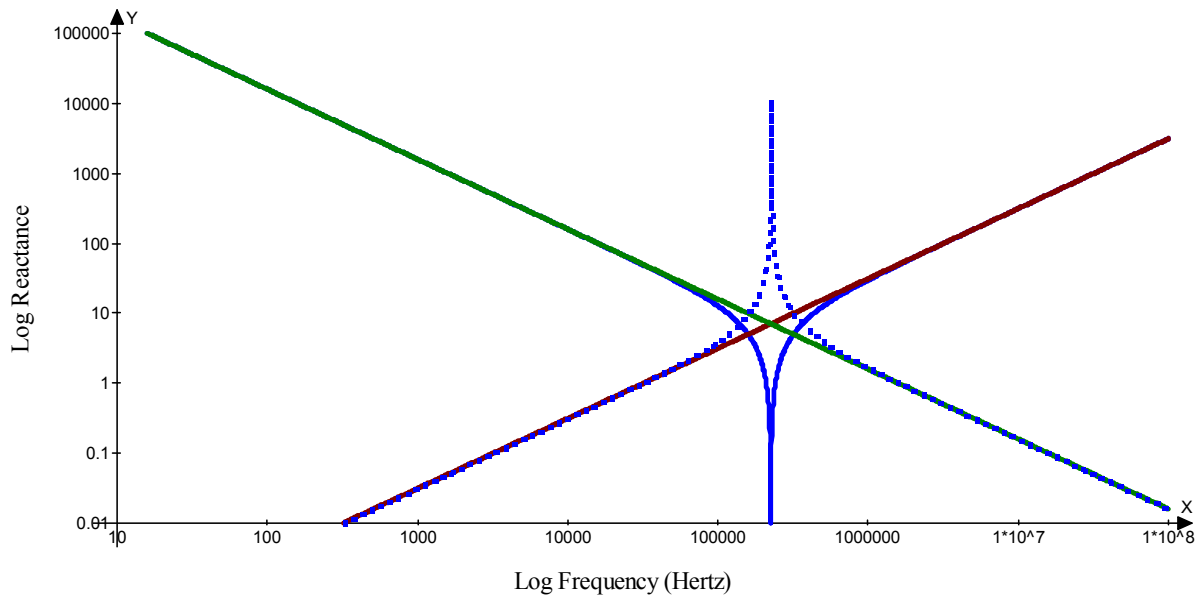

Figure 4
The effect of combining the capacitor and the inductor in series (blue)
or in parallel (dotted blue).

Note that in each case the blue curve follows (almost exactly) the green or red curves except near the point where the green and red curves intersect. That is, for a series combination, the curve is downward sloping to the right (capacitive) at lower frequencies and upward sloping to the right (inductive) at higher frequencies. For a parallel combination the dotted blue curve is upward sloping to the right (inductive) at lower frequencies and downward sloping to the right (capacitive) at higher frequencies. That is because in the case of the series combination, the capacitor tends to *block* the lower frequencies while the inductor tends to *block* the higher frequencies. In the case of the parallel combination, the capacitor tends to *pass* the higher frequencies and the inductor tends to *pass* the lower frequencies.

What is more interesting is what happens in the area where the red and the green curves cross. At that point (frequency) the reactance of the capacitor is equal to the reactance of the inductor, except that the phase difference is 180 degrees (the capacitor has a -90 degree phase shift and the inductor has a +90 degree phase shift.). So, they exactly cancel.

Look at the expression $1- \omega^2 LC$ and/or $\omega^2 LC-1$ that occurs in Equations 16 and 17. Let's set that expression equal to zero, or:

$$\omega^2 LC = 1$$
$$\omega^2 = 1/LC = 1/(5*10^{-6}*(0.1)*10^{-6}) = (2* \pi *f)^2$$
$$f = 318 \text{ KHz}$$

This frequency is called the resonant frequency. For a series combination of L and C, the resonant frequency is where the numerator of the reactance function does to zero and the reactance is ZERO --- not just low, not just very small, but ZERO. In the case of a parallel combination, it is the denominator that goes to zero and the reactance becomes INFINITE at the resonant frequency --- not just high, not just really high, but INFINITE. The only way to have some real value here, instead of zero or infinity, at the resonant frequency is to have some resistance in the circuit, in which case it is no longer purely reactive (see the case of Impedance in Part 3.) Series or parallel combinations of L and C are commonly used in filter circuits because you can select a specific frequency to pass or block *very* precisely.

For a much more extensive discussion of what happens when we combine reactances, see Brooks, Note 9.

Power Considerations:

In Part 1 of this series we discussed how resistance dissipates power, and how i^2R heating causes traces to heat up (among other things.) There is no comparable dissipation with reactance. Capacitors and inductors absorb energy in creating the electric and magnetic fields when current flows. But when these fields collapse, *all* the energy (at least for *pure* reactances) is returned to the circuit. Energy can only be lost if there is some resistance present.

There are, of course, some qualifiers to this statement. Consider a transmitter circuit where the circuit is driving an antenna. The antenna radiates electromagnetic energy out into space. That energy is lost to the circuit. The circuit acts as if there were some resistive loss, and indeed the equivalent circuit includes a virtual resistance called "radiation resistance (see Note 10)." Energy and power are lost to the circuit, not in the form of heat, but in the form or radiated electromagnetic energy. Nevertheless, power, in the form of heat, is not lost in a purely reactive circuit.

Notes:
1. Brooks has an extensive discussion of the three components in Chapters 5 and 6 of his book, <u>Signal Integrity Issues and Printed Circuit Board Design</u>, Prentice Hall, 2003.
2. See in particular Brooks, "What's This Thing Called 'Current'? *Electrons, Displacement, Light or What?"*   available <u>http://www.ultracad.com/mentor/what_is_current.pdf</u>
3. Michael Faraday, (1791 – 1867)
4. It is named after Joseph Henry (1797-1878), the American scientist who discovered electromagnetic induction independently of and at about the same time as Michael Faraday (1791-1867) in England.
5. You may have heard that traces should be routed close to a related, continuous plane in order to reduce loop inductance and reduce EMI. The reason that routing traces close to a related underlying plane reduces inductance is that the signal (on the trace) and the return (on the plane) are equal and opposite and their magnetic effects tend to cancel each other out.
6. Appendix 5 in Brooks' book (see Note 1) is devoted to explaining complex algebra and related operations.
7. Paul Nahin has written a history of the thought surrounding the imaginary operator. See Paul J. Nahin, <u>An Imaginary Tale: The Story of *i* [the square root of minus one]</u>, Princeton University Press, 1998
8. Named after the German physicist Heinrich Hertz. See <u>http://en.wikipedia.org/wiki/Hertz</u>.
9. See Brooks' book, Note 1, Chapter 6, "Reactance", pages 97-119
10. See, for example, <u>http://en.wikipedia.org/wiki/Radiation_resistance</u> .